1a) Formulate the principle of mathematical induction.

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flawless, my boy

I'm so proud :)

Let S_n be a statement about the positive integer n. Suppose that:

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1) S_1 is true
2) $\forall n > 0$ S_{n+1} is true whenever S_n is true.

Then S_n is true for every positive integer n.

1b) Use the principle of mathematical induction to show that $3^n - 1$ is even for every natural number $n \geq 1$.

Statement: $3^n - 1$ is even for every natural number $n \geq 1$.

Proof by mathematical induction:

1) Base step: fill in $n = 1$

$S_1 = 3^n - 1 = 3^1 - 1 = 3 - 1 = 2 \Rightarrow$ 2 is even, so S_1 holds.

2) Induction step: assume that S_n is true, prove that S_{n+1} is true

$S_n = 3^n - 1$ (which is even by assumption, where $n \geq 1$)

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$$
\begin{aligned}
S_{n+1} &= 3^{n+1} - 1 \qquad \text{where } n \geq 1 \\
&= 3 \cdot 3^n - 1 \\
&= 3 \cdot (3^n - 1) + 2 \\
&= 3(S_n) + 2
\end{aligned}
$$

we assumed that $3^n - 1$ is even, so $3(3^n - 1)$ must be even too. Because 2 is also even, S_{n+1} has to be even.

Thus, by using mathematical induction, we have proven that $3^n - 1$ is even for every natural number $n \geq 1$.

Q.E.D.

2) Use mathematical induction to show that $r^0 + r^1 + r^2 + \ldots + r^{n-1} = \frac{r^n - 1}{r-1}$ for all integers $n \geq 1$ and $r \neq 1$.

Statement: $r^0 + r^1 + r^2 + \ldots + r^{n-1} = \frac{r^n - 1}{r-1}$ where $n \geq 1$, $n \in \mathbb{N}$ and $r \neq 1$

Proof by mathematical induction:

1) Base step: fill in $n=1$

$$\left.\begin{aligned} S_1 &= r^{n-1} = r^{1-1} = r^0 = 1 \\ S_1 &= \frac{r^n - 1}{r-1} = \frac{r^1 - 1}{r-1} = \frac{r-1}{r-1} = 1 \end{aligned}\right\}$$ $1=1$, so S_1 holds ✓

2) Induction step: assume that S_n is true, prove that S_{n+1} is true.

S_n: $r^0 + r^1 + r^2 + \ldots + r^{n-1} = \frac{r^n - 1}{r-1}$ where $n \geq 1$, $n \in \mathbb{N}$ and $r \neq 1$

S_{n+1}: $\underbrace{r^0 + r^1 + r^2 + \ldots + r^{n-1}} + r^{(n+1)-1} = \frac{r^{(n+1)} - 1}{r-1}$ where $n \geq 1$, $n \in \mathbb{N}$ and $r \neq 1$

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$$\frac{r^n - 1}{r-1} + r^{(n+1)-1} = \frac{r^{(n+1)} - 1}{r-1}$$

$$\frac{r^n - 1}{r-1} + r^n = \frac{r \cdot r^n - 1}{r-1}$$

$$\frac{r^n - 1}{r-1} + \frac{r^n(r-1)}{r-1} = \frac{r \cdot r^n - 1}{r-1}$$

$$\frac{r^n - 1 + (r \cdot r^n - r^n)}{r-1} = \frac{r \cdot r^n - 1}{r-1}$$

$$\frac{r^n - 1 + r \cdot r^n - r^n}{r-1} = \frac{r \cdot r^n - 1}{r-1}$$

$$\frac{r \cdot r^n - 1}{r-1} = \frac{r \cdot r^n - 1}{r-1}$$

$$\frac{r^{(n+1)} - 1}{r-1} = \frac{r^{(n+1)} - 1}{r-1}$$

These last terms are equal, so S_{n+1} holds.

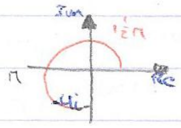
Thus, by using mathematical induction, we have proven that the statement $r^0 + r^1 + r^2 + \ldots + r^{n-1} = \frac{r^n - 1}{r-1}$ is true for all integers $n \geq 1$ and $r \neq 1$.

Q.E.D.

3) Solve $z^4 = \frac{4}{i}$ and sketch the solutions in the complex plane.

$$z^4 = \frac{4}{i} = \frac{4}{i} \cdot \frac{i}{i} = \frac{4i}{i^2} = \frac{4i}{-1} = -\frac{4i}{1} = -4i$$

This number $-4i$ can be written in the $z = r \cdot e^{i\theta}$ form:



$r(z^4) = \sqrt{a^2+b^2} = \sqrt{0^2+(-4)^2} = \sqrt{16} = 4$

$\theta(z^4) = 1\frac{1}{2}\pi + 2h\pi$ where $h = 0, 1, 2, 3 \ldots$

$\Rightarrow \quad z^4 = 4 \cdot e^{i(1\frac{1}{2}\pi + 2h\pi)}$ where $h = 0, 1, 2, 3, \ldots$

$z = \left(4e^{i(1\frac{1}{2}\pi + 2h\pi)}\right)^{\frac{1}{4}}$

$z = \sqrt[4]{4} \cdot e^{i((\frac{6}{4} \cdot \frac{1}{4})\pi + \frac{2}{4}h\pi)}$

$z = \sqrt{2} \cdot e^{i(\frac{6}{16}\pi + \frac{1}{2}h\pi)} = \sqrt{2} \cdot e^{i(\frac{3}{8}\pi + \frac{4}{8}h\pi)}$

Using this formula, one can find the following solutions:

$z_0 = \sqrt{2} \cdot e^{i(\frac{3}{8}\pi)}$ for $h = 0$

$z_1 = \sqrt{2} \cdot e^{i(\frac{7}{8}\pi)}$ for $h = 1$

$z_2 = \sqrt{2} \cdot e^{i(\frac{11}{8}\pi)} = \sqrt{2} \cdot e^{i(1\frac{3}{8}\pi)}$ for $h = 2$

$z_3 = \sqrt{2} \cdot e^{i(\frac{15}{8}\pi)} = \sqrt{2} \cdot e^{i(1\frac{7}{8}\pi)}$ for $h = 3$

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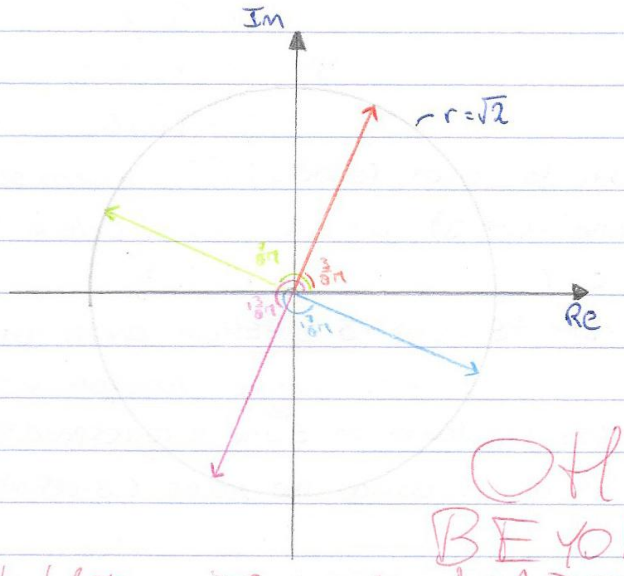
Flawless.

Sketching these numbers gives:



– : $z_0 = \sqrt{2}e^{i(\frac{3}{8}\pi)}$

– : $z_1 = \sqrt{2}e^{i(\frac{7}{8}\pi)}$

– : $z_2 = \sqrt{2}e^{i(1\frac{3}{8}\pi)}$

– : $z_3 = \sqrt{2}e^{i(1\frac{7}{8}\pi)}$

OH MY GOD IT'S BEYOND PRETTY

THIS JUST MADE MY DAY

thank you. Colours. Just... thank you.

4) Determine all complex numbers z satisfying $e^{iz} = e^{-z}$

$$e^{iz} = e^{-z}$$

$$e^{iz} = \frac{1}{e^{z}}$$

$$e^{iz} \cdot e^{z} = 1$$

1 can be written as: $e^{i(0+2k\pi)}$

$$e^{(i+1)z} = 1 \cdot e^{i(0+2k\pi)}$$

where $z = a + bi$ and $k = 1, 2, 3, \ldots$

$$e^{(i+1)(a+bi)} = 1 \cdot e^{i(0+2k\pi)}$$

$$e^{(ai+a-b+bi)} = 1 \cdot e^{i(0+2k\pi)}$$

$$e^{a-b} \cdot e^{ai+bi} = 1 \cdot e^{i(0+2k\pi)}$$

$\Rightarrow \quad e^{a-b} = 1 \Rightarrow a - b = 0 \Rightarrow \underline{a = b}$

$e^{ai+bi} = e^{i(a+b)} = e^{i(2k\pi)} \Rightarrow a + b = 2k\pi$

$b + b = 2k\pi$

$b = k\pi = a$

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So: $z = a + bi$

$\underline{\underline{= k\pi + k\pi i}}$ where $k = 1, 2, 3, \ldots$

5) Prove - using the precise ε,δ-definition of a limit - that: $\lim_{x \to 3} x^2 = 9$

Given: $\lim_{x \to a} f(x) = L$ $\quad \forall \varepsilon > 0 \; \exists \delta > 0$ s.t. $0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon$ (general)

$\lim_{x \to 3} x^2 = 9$ $\quad \forall \varepsilon > 0 \; \exists \delta > 0$ s.t. $0 < |x - 3| < \delta \Rightarrow |x^2 - 9| < \varepsilon$ (specific)

We want to link ε and δ. So, we have to rewrite $|x^2 - 9| < \varepsilon$

$$|x^2 - 9| < \varepsilon$$

$$|(x+3)(x-3)| < \varepsilon$$

$$|x - 3| < \frac{\varepsilon}{x+3}$$

$$|x - 3| < \frac{\varepsilon}{c}$$

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We can find the (maximum) value for c as follows:

$2 < x < 4$ (because $x \to 3$)

$5 < x + 3 < 7 \quad \Rightarrow \quad c = 7$

This gives: $|x - 3| < \frac{\varepsilon}{7} \Rightarrow |x - 3| < \frac{1}{7}\varepsilon \Rightarrow \underline{\delta = \frac{1}{7}\varepsilon}$.

The condition of our proof is satisfied (we found an $\varepsilon > 0$ and a corresponding $\delta > 0$).
Thus, we have proven that $\lim_{x \to 3} x^2 = 9$ by using the precise ε,δ-definition.

Q.E.D.